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NEW ALGEBRAIC CRITERIA FOR POSITIVE REALNESS*

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ABSTRACT

In this work, we will present new algebraic criteria for positive realness of real rational functions and matrices, which are formulated entirely in terms of the Routh-Hurwitz conditions. It is first shown how the Routh algorithm can be modified to serve as a criterion for positive realness. Due to the outstanding analytic simplicity of the Routh algorithm, this criterion yields an efficient numerical procedure for testing positive realness. Once the Routh algorithm is successfully reformulated, the Hurwitz version of positive realness is almost automatic. Hurwitz-like determinants are obtained, which provide explicit conditions for positive realness often desired in the theory of networks and systems. The matrix generalization of the proposed Routh-Hurwitz criteria leads to numerical procedures for testing the positive real character of real rational matrices with desirable simplicity and suitability for machine calculations.

Introduction

Since the concept of positive real functions was introduced by Brune [1] in 1931, it not only laid the basis for all of the realization theory of electrical networks [2], but was subsequently utilized in areas as diverse as absolute stability and hyperstability, optimality and sensitivity of dynamic systems. The appeal to positive real function formulation in these areas was based largely upon the physical insight offered by the positive real concept as well as the mathematical compactness of the formulation. In applications, however, the usefulness of the positive real concept ultimately depends on the efficiency of the numerical procedure for testing the positive real character of a given function or matrix representing a network or system of a considerable complexity. For obvious reasons, the procedure should lead to systematic algebraic schemes which can readily be programmed for computer applications.

As known [2], the algebraic testing of the positive real character of a rational function involves two polynomial algorithms, namely the Routh array and the Sturm method, which are performed by two essentially different schemes [3]. While the Routh array is executed by a simple recursive process, the Sturm method requires a construction of Sturm's functions by repeated polynomial divisions which are inconvenient for numerical calculations. Furthermore, certain nonsystematized additional analysis is required in the Sturm method when polynomials to be divided are not relatively prime. It is surprising that only recently it has been shown [4] how the well-known Routh array can be modified to carry out the required Sturm's procedure and, therefore, perform the entire positive real test by using the Routh algorithm.

In this work, we will give a complete account of the modified Routh algorithm for positive realness and solve the important problem of singular cases which was only briefly indicated in [4]. This will lead to a recursive, computer oriented scheme with superior effectiveness over the present numerical procedures for testing the positive real character of real rational functions. An important by-product of this result is a solution of the classical algebraic problem: the formulation of necessary and sufficient conditions on the coefficients of a real polynomial under which the polynomial has a certain number of real positive zeros.

Another important aspect of this work lies in the formulation of the general explicit conditions for positive realness. It is remarkable that the derived determinantal conditions are analogous to the celebrated Hurwitz conditions. That such conditions are not available despite a relatively long history of the positive real concept is probably due to the automatic use of the Sturm method which obscures the issue and discourages any attempt in this direction.

Extension of the obtained criteria to the testing of real rational matrices is straightforward and is derived for the scattering matrix formulation for obvious practical reasons. It is certainly here that the proposed new test is by far the most efficient scheme among the existing numerical procedures for testing the positive real character of real rational matrices.

Fundamental Algebraic Problem

We start with a definition of the positive real function [5]:

Definition 1: A real rational function

$$G(s) = \frac{q(s)}{p(s)} \tag{1}$$

with relatively prime polynomials p(s) *and* q(s) *is called positive real if and only if:*

(i) *the polynomial*

$$f(s) = p(s) + q(s) \tag{2}$$

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is Hurwitz; and

(ii) Re $G(i\omega) \geq 0$, *for all real* ω. (3)

Remark 1: If (3) is replaced by the strict inequality

$$\text{Re } G(i\omega) > 0, \text{ for all real } \omega, \quad (4)$$

the function $G(s)$ is said to be *strictly positive real*.

As is customary, we call a function $G(s)$ real if the polynomials $p(s)$ and $q(s)$ have real coefficients, that is, they are real polynomials. Furthermore, two polynomials are relatively prime if they have no common cancellable factors, and a polynomial is said to be Hurwitz if it has all zeros with negative real parts.

To establish the positive real character of a given function, one applies the Routh or Hurwitz test to verify (i) and Sturm's procedure to examine (ii) of Definition 1. The Sturm procedure is actually used to test the polynomial inequality

$$g(\omega^2) \equiv p_r(\omega)q_r(\omega) + p_i(\omega)q_i(\omega) \geq 0, \quad \text{for all real } \omega \geq 0 \quad (5)$$

with $p(i\omega) = p_r(\omega) + ip_i(\omega)$ and $q(i\omega) = q_r(\omega) + iq_i(\omega)$, which is obviously equivalent to (3). The polynomial $g(\omega^2)$ is real and even. We assume that the polynomials in (1) are given as

$$p(s) \equiv \sum_{u=0}^{m} p_u s^u, \quad q(s) \equiv \sum_{v=0}^{\ell} q_v s^v \quad (6)$$

where $m \geq \ell$, and some of the coefficients p_u and q_v may be missing but not p_m. Since $G(s)$ in (1) is positive real if and only if $G^{-1}(s)$ is positive real, when $m < \ell$ we examine $G^{-1}(s)$ instead of $G(s)$. Therefore, (5) can be written as

$$g(\omega^2) \equiv \sum_{k=0}^{n} g_{2k}\omega^{2k} \geq 0, \text{ for all real } \omega \geq 0, \quad (7)$$

where $2n = [\text{even } (m + \ell)]$, which means that $2n$ is equal to the largest even integer contained in $m + \ell$, and

$$g_{2k} = \sum_{\tau=0}^{2k} (-1)^{k+\tau} p_\tau q_{2k-\tau}, \quad k = 0, 1, \ldots, n. \quad (8)$$

In formula (8), the coefficients which are missing in (6) are replaced by zeros.

From (7), we have the obvious preliminary result:

Lemma 1: *Polynomial* $g(\omega^2) \geq 0$ *for all real* $\omega \geq 0$ *if and only if* $g(\omega^2) \not\equiv g_0$ *has no positive real zeros of odd multiplicity and* $g(\omega_1^2) > 0$ *for some* $\omega_1 \geq 0$; *or* $g(\omega^2) \equiv g_0 \geq 0$.

Remark 2: Condition $g(\omega_1^2) > 0$ for some $\omega_1 \geq 0$ can be replaced by either $g_0 > 0$ (that is, $g(0) > 0$), or $g_{2n} > 0$ (that is, $g(+\infty) > 0$), which are easy to verify. In the following, when $g_0 = g_2 = \ldots = g_{2i} = 0$, we can write $g(\omega^2) \equiv \omega^{2i+2} g_1(\omega^2)$ and consider the reduced polynomial $g_1(\omega^2)$.

In case of the strict positive realness (Remark 1), we will need:

Lemma 2: *Polynomial* $g(\omega^2) \geq 0$ *for all real* $\omega \geq 0$ *if and only if* $g(\omega^2)$ *has no real positive zeros and* $g(\omega_1^2) > 0$ *for some* $\omega_1 \geq 0$.

Therefore, the part (ii) of the above definition is reduced to the problem of finding the necessary and sufficient conditions under which a real even polynomial has no real positive zeros of odd multiplicity or none at all. In the following section, we will first solve the general problem: *determine the necessary and sufficient conditions on the coefficients of a given real polynomial so that the polynomial has a certain number of positive real zeros.* The solution of this problem will be given by the Routh algorithm which is modified as suggested in [6]. Then, we will extend the obtained result to the even polynomials.

For now, it may be of interest to mention the following trivial, but useful result [7] which is derived from (7) and the well-known Decartes rule of signs:

Theorem 1: *Polynomial* $g(\omega^2) > 0$ *for all real* $\omega \geq 0$:

(i) *if*

$$g_0 > 0, \; g_{2k} \geq 0, \; k = 1, 2, \ldots, n; \quad (9)$$

and (ii) *only if the total number of sign variations in the coefficients* g_{2k} $(k = 0, 1, \ldots, n)$ *is even.*

Inequalities (9) may be utilized when coefficients g_{2k} depend on parameters and one is interested to find the region in the corresponding parameter space such that $g(\omega^2) > 0$ for all real $\omega \geq 0$. [6]

Modified Routh Algorithm

We first prove our general result:

Theorem 2: *The number* π *of distinct positive real zeros of the real polynomial*

$$h(s) \equiv \sum_{k=0}^{n} h_k s^k, \; h_n \neq 0 \quad (10)$$

is

$$\pi = n - V[(-1)^n h_n, (-1)^n n h_n, \ldots, h_0] \quad (11)$$

where V *is the number of sign variations in the first column of the Routh array*

$$\begin{array}{lllll} (-1)^n h_n & (-1)^{n-1} h_{n-1} & \cdots & -h_1 & h_0 \\ (-1)^n n h_n & (-1)^{n-1}(n-1) h_{n-1} & \cdots & -h_1 & \\ \vdots & \vdots & & & \\ h_0 & & & & \end{array} \quad (12)$$

and $h_0 \neq 0$.

Remark 3: We note that the first two rows of the Routh array (12) are formed by the coefficients of the polynomials $h(-s)$ and $h'(-s) \equiv dh(-s)/ds$.

To prove Theorem 2, we start with the well-known Sturm theorem (Theorem 1, Chapter XV, Reference 3):

Theorem 3: *The number* π *of the distinct positive real zeros of a real polynomial* $h(s)$ *is*

$$\pi = I_0^{+\infty} \frac{h'(s)}{h(s)} \tag{13}$$

where I *denotes the Cauchy index and* $h'(s) \equiv dh(s)/ds$.

By using the properties of Cauchy indicies [3] we will transform expression (13) until it can be evaluated by the Routh algorithm. A hint for this transformation lies in a theorem of Stieltjes (Theorem 15, Chapter XV, Reference 3).

We assume that $h_0 \neq 0$, and rewrite (13) as

$$\pi = I_0^{+\infty} \frac{h'(s)}{h(s)} = I_0^{+\infty} \frac{\omega h'(\omega^2)}{h(\omega^2)}$$

$$= \frac{1}{2} [I_{-\infty}^{0} \frac{\omega h'(\omega^2)}{h(\omega^2)} + I_0^{+\infty} \frac{\omega h'(\omega^2)}{h(\omega^2)}]$$

$$= \frac{1}{2} I_{-\infty}^{+\infty} \frac{\omega h'(\omega^2)}{h(\omega^2)} = \frac{1}{2} I_{-\infty}^{+\infty} \frac{\sum_{k=1}^{n} k\, h_k\, \omega^{2k-1}}{\sum_{k=1}^{n} h_k\, \omega^{2k}} \tag{14}$$

Following Routh, we compute the Cauchy index in the last equation of (14) by the array (12) and obtain (11). This proves Theorem 2.

Remark 4: The *regular* Routh array takes place when none of the numbers in the first column of the array vanish. A *singular* array arises when: (a) an element of the first column becomes zero, but not all of the numbers in the corresponding row are zero; and (b) all the numbers in a row of the array vanish simultaneously. The case (a) is of no importance and is resolved by standard techniques [3]. The case (b) indicates that $h(s)$ and $h'(s)$ have common zeros, that is, they are not relatively prime which means that $h(s)$ has *multiple zeros*. We postpone the consideration of the multiple zeros to the next section. For now, we recall [3] that to continue the Routh algorithm in case (b), it is necessary to replace the row of zeros by the coefficients of the first derivative of the polynomial formed by the preceeding row. We also note that the assumption $h_0 \neq 0$ means no loss of generality since when $h_0 = h_1 = \ldots = h_i = 0$, we can write $h(s) \equiv s^{i+1} h_1(s)$ and consider the reduced polynomial $h_1(s)$.

Remark 5: Obviously, to determine the number of negative real zeros of $h(s)$, we apply Theorem 2 to the polynomial $h(-s)$.

In case of the real even polynomial $g(\omega^2)$ of (7), we replace ω^2 by ω and form the Routh array:

$$\begin{array}{lllll} (-1)^n g_{2n} & (-1)^{n-1} g_{2n-2} & \cdots & -g_2 & g_0 \\ (-1)^n n g_{2n} & (-1)^{n-1}(n-1) g_{2n-2} & \cdots & -g_2 & \\ \vdots & \vdots & & & \\ g_0 & & & & \end{array} \tag{15}$$

which was used in [4].

By applying Theorem 2 and the array (15) with Remark 4, we get:

Corollary 1: *The number* π *of the positive real zeros of* $g(\omega^2)$ *is*

$$\pi = n - V[(-1)^n g_{2n}, (-1)^n n g_{2n}, \ldots, g_0], \tag{16}$$

where V *is the number of sign variations in the first column of the array* (15).

We need immediately a more specific:

Corollary 2: *A real even polynomial* $g(\omega^2)$ *of degree* 2n *has no positive zeros* ($\pi = 0$) *if and only if*

$$V[(-1)^n g_{2n}, (-1)^n n g_{2n}, \ldots, g_0] = n. \tag{17}$$

Remark 6: Since the polynomial $g(\omega^2)$ is even and with real coefficients, the real zeros appear in pairs and Corollary 1 yields immediately both the real positive and real negative zeros (if such zeros exist, Corollary 2).

Before we state our main result concerning the positive real functions, let us recall the original Routh test [3] used to test the Hurwitz character of the polynomials with real coefficients. This is needed for part (i) of Definition 1. Let us consider the polynomial $f(s)$ in (2) written as

$$f(s) \equiv \sum_{w=0}^{m} f_w s^w, \quad f_m \neq 0, \tag{18}$$

where, according to (6), we have

$$f_w = p_w + q_w, \quad w = 0, 1, \ldots, m, \tag{19}$$

$p_m \neq 0$, and $q_w \equiv 0$ for $w > \ell$. As known [3], the polynomial $f(s)$ is Hurwitz if and only if in the corresponding Routh array:

for m even / *for m odd*

$$\begin{array}{lllll} f_m & f_{m-2} & \cdots & f_2 & f_0 \\ f_{m-1} & f_{m-3} & \cdots & f_1 & \\ \vdots & \vdots & & & \\ f_0 & & & & \end{array} \qquad \begin{array}{lllll} f_m & f_{m-2} & \cdots & f_2 & f_0 \\ f_{m-1} & f_{m-3} & \cdots & f_3 & f_1 \\ \vdots & \vdots & & & \\ f_0 & & & & \end{array} \tag{20}$$

all elements of the first column are different from zero and of like sign, that is,

$$V[f_m, f_{m-1}, \ldots, f_0] = 0 \tag{21}$$

Now, by using Corollary 2, Lemma 2, and Remark 4, we obtain directly:

Theorem 4: *A real rational function* $G(s) = q(s)/p(s)$ *with relatively prime polynomials* $p(s)$ *and* $q(s)$ *is strictly positive real if and only if:*

(i) *the polynomial* $f(s)$ *produces a regular Routh array* (20) *with*

$$V[f_m, f_{m-1}, \ldots, f_0] = 0; \tag{21}$$

and

(ii) *the polynomial* $g(\omega^2)$ *produces a Routh array* (15) *with*

$$V[(-1)^n g_{2n}, (-1)^n n g_{2n}, \ldots, g_0] = n \tag{17}$$

and $g_0 > 0$.

This result was obtained in [4] and is considerably simpler than the one derived in [9], since the modified Routh array (15) proposed here operates directly on the coefficients of polynomial $g(\omega^2)$ and no intermediate steps are necessary.

For nonstrict positive realness of $G(s)$, we need to test the weaker polynomial inequality (7) which allows for positive real zeros of $g(\omega^2)$ so far they are of even multiplicity (Lemma 1). We will show immediately that the Routh array (15) contains all the necessary information about multiple real zeros of the polynomial $g(\omega^2)$ and that a recursive algorithm for determining those multiplicities is available.

We remember first that the multiple zeros of $g(\omega^2)$ are indicated by occurrence of zero rows in the corresponding Routh array (Remark 4). Let us enumerate the rows in the array (15) starting from the top by j $(j = 0, 1, \ldots, 2n)$ and the row preceeding a zero row by j_ν $(\nu = 1, 2, \ldots, t)$. Let us also define

$$n_\nu = \frac{1}{2}(j_\nu - j_{\nu-1}), \quad \nu = 1, 2, \ldots, t+1 \tag{22}$$

with $j_0 = 1$ and $j_{t+1} = 2n + 1$. For the number of sign variations between the two consecutive rows $j_{\nu-1}$ and j_ν, we will use the symbol V_ν. Finally, we denote by π_ν the number of positive real zeros the multiplicity ν of the polynomial $g(\omega^2)$. The number π_ν we determine using the general recurrence formula

$$\pi_\nu = (n_\nu - V_\nu) - (n_{\nu+1} - V_{\nu+1}), \quad \nu = 1, 2, \ldots, t+1. \tag{23}$$

By applying Corollary 1, the formula (23) can be readily verified. We need only to notice that π_ν is actually the difference between the numbers of positive real zeros corresponding to two consecutive zero rows in array (15), and that the multiplicity of each zero of the tested polynomial is reduced by one each time we go from one zero row to the next.

Now, according to Definition 1, Lemma 1, and formula (23), we have:

Theorem 5: *A real rational function* $G(s) = q(s)/p(s)$ *with relatively prime polynomials* $p(s)$ *and* $q(s)$ *is positive real if and only if:*

(i) *all the elements in the first column of the Routh array* (20) *produced by the polynomial* $f(s)$ *are different from zero and of like sign; and*

(ii) *all the numbers* π_ν *produced by the polynomial* $g(\omega^2) \not\equiv g_0$ *and the Routh array* (15) *are equal to zero for odd* $\nu = 1, 3, \ldots, [\text{odd } (t+1)]$, *where* t *is the number of zero rows in array* (15), *and* $g(\omega_1^2) > 0$ *for some* $\omega_1 \geq 0$; *or* $g(\omega^2) \equiv g_0 \geq 0$.

In (ii) above, [odd $(t + 1)$] means the largest integer contained in $(t + 1)$.

Remark 7: If there are no zero rows $(t = 0)$ in array (15), then Theorem 5 is reduced to Theorem 4. In this case, $\pi_1 = n_1 - V_1$, where $n_1 = n$ and $V_1 = V$ (see Corollary 1).

Remark 8: Since recurrence formula (23) involves only two consecutive rows at the time, the test may be discontinued when we first encounter $\pi_\nu \neq 0$ for odd ν.

Remark 9: As shown in [10], if we form the first two rows of the Routh array by the coefficients of $p(s)$ and $q(s)$, then, $p(s)$ and $q(s)$ are relatively prime if and only if there is no rows in the array that vanish identically.

It is remarkable that with results obtained in this section all one needs in testing positive realness is the Routh Algorithm.

Modified Hurwitz Criterion

An alternative way to compute the Cauchy index in (14) is by the Hurwitz approach [3], which yields explicit conditions for positivity of $G(s)$ in terms of its coefficients. The Hurwitz-like solution is more appealing in cases where the coefficients are not given numerically but depend on parameters.

By using the Routh array (15), we can formally generate the desired determinants:

$$\Delta_1 = \left|(-1)^n n g_{2n}\right|$$

$$\Delta_2 = \begin{vmatrix} (-1)^n n g_{2n} & (-1)^{n-1}(n-1) g_{2n-2} \\ (-1)^n g_{2n} & (-1)^{n-1} g_{2n-2} \end{vmatrix}$$

$$\vdots \tag{24}$$

$$\Delta_{2n} = \begin{vmatrix} (-1)^n n g_{2n} & (-1)^{n-1}(n-1) g_{2n-2} & \cdots & 0 \\ (-1)^n g_{2n} & (-1)^{n-1} g_{2n-2} & \cdots & 0 \\ 0 & (-1)^n n g_{2n} & \cdots & 0 \\ 0 & (-1)^n g_{2n} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ & & \cdots & g_0 \end{vmatrix}$$

which represent the first column of the array (15). Since $\Delta_{2n} = g_0 \Delta_{2n-1}$, by using Corollary 1 we arrive at:

Theorem 6: The number π of the positive real zeros of $g(\omega^2)$ is given by

$$\pi = n - V[(-1)^n g_{2n},\ \Delta_1,\ \Delta_2/\Delta_1,\ldots,\ \Delta_{2n-1}/\Delta_{2n-2},\ g_0] \tag{25}$$

provided $\Delta_j \neq 0$, $j = 1, 2,\ldots,2n-1$, *and* $g_0 > 0$.

This theorem can readily be proved by the triangularization of the matrix corresponding to Δ_{2n} which reduces it to the Routh matrix defined by (15). This process is described in [3].

Remark 10: When some of the determinants in (25) are equal to zero (singular case [3], but $\Delta_{2n-1} \neq 0$, in calculation of V for each group of λ (odd) successive zero determinants

$$(\Delta_\mu \neq 0) \quad \Delta_{\mu+1} = \ldots = \Delta_{\mu+\lambda} = 0 \ (\Delta_{\mu+\lambda+1} \neq 0) \tag{26}$$

we have to set:

$$V(\Delta_\mu/\Delta_{\mu-1},\ \Delta_{\mu+1}/\Delta_\mu,\ldots,\ \Delta_{\mu+\lambda+2}/\Delta_{\mu+\lambda+1})$$

$$= \frac{1}{2}\,[\lambda+2 - (-1)^{(\lambda+1)/2}\ \mathrm{sign}(\Delta_\mu\Delta_{\mu+\lambda+2}/\Delta_{\mu-1}\Delta_{\mu+\lambda+1})], \tag{27}$$

as shown in [3].

For the sake of completeness, let us consider again the polynomial f(s) in (18) and recall [3] that the polynomial is Hurwitz if and only if

$$V(f_m,\ \delta_1,\ \delta_2/\delta_1,\ldots,\ \delta_m/\delta_{m-1}) = 0 \tag{28}$$

where $f_m > 0$. In (28),

$$\delta_\alpha = \det\,[h_{\beta\gamma}]\ ,\ \alpha = 1, 2,\ldots, m \tag{29}$$

are principal minors of the $m \times m$ Hurwitz matrix $H = [h_{\beta\gamma}]$ whose elements are given by

$$h_{\beta\gamma} = f_{m+\beta-2\gamma}\ , \tag{30}$$

and $f_w \equiv 0$ for $w < 0$ or $w > m$.

Now, by Theorem 6, Remark 10, and (28) we obtain:

Theorem 7: A real rational function $G(s) = q(s)/p(s)$ with relatively prime polynomials $p(s)$ and $q(s)$ is strictly positive real if and only if:

(i) $V(f_m,\ \delta_1,\ \delta_2/\delta_1,\ldots,\ \delta_m/\delta_{m-1}) = 0$ (28)

where $f_m > 0$, $\delta_{m-1} \neq 0$; *and*

(ii)

$$V[(-1)^n g_{2n},\ \Delta_1,\ \Delta_2/\Delta_1,\ldots,\ \Delta_{2n-1}/\Delta_{2n-2},\ g_0] = n \tag{31}$$

where $g_0 > 0$, $\Delta_{2n-1} \neq 0$.

Remark 11: if all the coefficients of f(s) are positive, the condition (28) can be reduced to

$$V(1,\ \delta_1,\ \delta_3,\ldots,\ \delta_{[\mathrm{odd}\ (m)]}) = 0 \tag{32}$$

which is the well-known Lienard-Chipart result [3].

As it was pointed out in [3], if the coefficients of the tested polynomial are given numerically, then the Routh algorithm is by far the simplist procedure for computing the Hurwitz determinants and executing the Sturm procedure. Therefore, when testing a real rational function with specified coefficients for possible positive realness, the results of the preceeding section are recommended.

Matrix Generalization

Extension of Theorem 5 to the matrix case is desirable since it promises a positive real test with a satisfactory analytic simplicity.

We start with [11]:

Definition 2: A real rational $m \times m$ matrix G(s) is positive real if and only if:

(i) *G(s) has no poles with positive real parts;*

(ii) *poles of G(s) on Re s = 0 are simple and the corresponding residue matrix in nonnegative definite Hermitian; and*

(iii) $G(i\omega) + G^*(i\omega) > 0$ *for all real ω such that $i\omega$ is not a pole of $G(i\omega)$.*

To avoid complications involved in testing the condition (ii) of Definition 2, we may perform certain matrix transformations and use the well-known [11]:

Theorem 8: A real rational matrix G(s) is positive real if and only if the corresponding real rational scattering matrix

$$S(s) = [G(s) - I][G(s) + I]^{-1} \tag{33}$$

is bounded real, that is,

(i) *S(s) is analytic in Re s $\geq$ 0; and*

(ii) $I - S^*(i\omega)S(i\omega) \geq 0$, *for all real ω*.(34)

Let us now consider the expression

$$[G(s) + I]^{-1} = \frac{F(s)}{f(s)} \tag{35}$$

where F(s) is a real polynomial $m \times m$ matrix and f(s) is a real scalar polynomial relatively prime to F(s). If we rewrite (33) as

$$S(s) = I - 2[G(s) + I]^{-1} = I - 2\,\frac{F(s)}{f(s)} \tag{36}$$

and use (34) to form the Hermitian $m \times m$ polynomial matrix

$$\Gamma(i\omega) \equiv |f(i\omega)|^2[1 - S^*(i\omega)S(i\omega)]\ , \tag{37}$$

we arrive immediately at:

Theorem : A real rational $m \times m$ matrix G(s) is positive real if and only if:

(i) *the polynomial* $f(s) \equiv \det\,[G(s) + I]$ *is Hurwitz; and*

(ii) $\Gamma(i\omega) \geq 0$, *for all real* $\omega \geq 0$. (38)

Let us assume that the matrix $\Gamma(i\omega)$ is of the rank r, that is, there is an r-th order principal minor of $\Gamma(i\omega)$ which is not identically zero and all the principal minors of order r + 1 vanish identically. We denote by $g^{(r)}(\omega^2)$ this r-th order minor, and by $g^{(\theta)}(\omega^2)$ the associated principal minors of order $\theta = 1, 2, \ldots, r - 1$. As is clear from (37), the minors are real even polynomials which can be written as

$$g^{(\theta)}(\omega^2) \equiv \sum_{k=0}^{n_\theta} g_{2k}^{(\theta)} \omega^{2k} . \qquad (39)$$

Now, to verify (38), we can use [5]:

Theorem 10: A Hermitian polynomial $m \times m$ *matrix* $\Gamma(i\omega)$ *of the rank* $r > 0$ *is nonnegative definite for all real* $\omega \geq 0$ *if and only if: (a) for* $r > 1$, *there is a sequence of the principal minors* $g^{(\theta)}(\omega^2) \not\equiv 0$, $\theta = 1, 2, \ldots, r - 1$, *such that*

$$g^{(\theta)}(\omega^2) \geq 0, \quad \textit{for all real } \omega \geq 0, \qquad (40)$$

and $\theta = 1, 2, \ldots, r$; *and (b) for* $r = 1$, $g^{(1)}(\omega^2) \geq 0$ *for all real* $\omega \geq 0$.

Remark 12: In the trivial case $r = 0$, each element of $\Gamma(i\omega)$ is identically zero and, therefore, $\Gamma(i\omega)$ is nonnegative definite for all real ω. In addition, we note that to test (38) by Theorem 10, we can use without loss of generality [5] the specific sequence of the principal minors $g^{(\theta)}(\omega^2)$, $\theta = 1, 2, \ldots, r = 1$, where each $g^{(\theta)}(\omega^2)$ is obtained from $g^{(r)}(\omega^2)$ by deleting the first $r - \theta$ rows and columns. The specific sequence is used in the following Theorem 11.

Finally, we can apply the modified Routh test to each inequality (40) and with help of Theorem 10 and Remark 12 get from Theorem 9 our principal result:

Theorem 11: A real rational $m \times m$ *matrix* $G(s)$ *is positive real if and only if:*

(i) *all the elements in the first column of the Routh array* (20) *produced by the polynomial* $f(s) \equiv \det [G(s) + I]$ *are different from zero and of like sign; and*

(ii) *all the numbers* $\pi_\nu^{(\theta)}$ *produced by the Routh array* (15) *and each principal minor* $g^{(\theta)}(\omega^2) \not\equiv g_0$, $\theta = 1, 2, \ldots, r$, *of the corresponding matrix* $\Gamma(i\omega)$, *are equal to zero for odd* $\nu = 1, 3, \ldots, [\mathrm{odd}(t_\theta + 1)]$, *where* r *is the rank of* $\Gamma(i\omega)$ *and* t_θ *is the number of the zero rows in the array* (15) *generated by* $g^{(\theta)}(\omega^2)$. *In addition,* $g^{(\theta)}(\omega_\theta^2) > 0$ *for some* $\omega_\theta \geq 0$ *and all* $\theta = 1, 2, \ldots, r$. *If* $g^{(\theta)}(\omega^2) \equiv g_0^{(\theta)}$, *then for* $r > 1$, $g_0^{(\theta)} > 0$, *and for* $r \leq 1$, $g_0^{(1)} \geq 0$.

We can actually avoid the matrix operations involved in forming the scattering matrix, if we wish to do so, and apply Theorem 11 directly to the matrix $G(s)$. This alternative is provided by:

Theorem 12: A real rational $m \times m$ *matrix* $G(s) = Q(s)/p(s)$ *with a real polynomial* $m \times m$ *matrix* $Q(s)$ *and a real scalar polynomial* $p(s)$ *relatively prime to* $Q(s)$ *is positive real if and only if:*

(i) *the polynomial* $f(s) = \det [G(s) + I]$ *is Hurwitz; and*

(ii) *the Hermitian* $m \times m$ *polynomial matrix* $\Lambda(i\omega) \equiv p^*(i\omega)Q(i\omega) + p(i\omega)Q^*(i\ )$ *satisfies the inequality*

$$\Lambda(i\omega) \geq 0, \ \textit{for all real } \omega \geq 0. \qquad (41)$$

If we rewrite (40) as

$|p(i\omega)|^2[G(i\omega) + G^*(i\omega)] \geq 0$, *for all real* $\omega \geq 0$ and note that

$$2[G(s) + G^*(s)] =$$

$$[I + G^*(s)][I - S^*(s)S(s)][I + G(s)], \qquad (42)$$

then from Theorem 8 the necessity of (i) and (ii) of Theorem 12 is automatic. Conversely, since $\det [G(s) + I] \not\equiv 0$, conditions (i) and (ii) of Theorem 12 imply those of Theorem 8. Hence, in Theorem 11, we can use $\Lambda(i\omega)$ instead of $\Gamma(i\omega)$.

An extension of this result to matrices that are functions of several complex variables is possible, which is of considerable interest in modern network theory.

Conclusion

The principal contribution of this work is a unique and complete recursive algorithm for testing the positive real properties of real rational functions and matrices. The algorithm is entirely based upon the familiar Routh-Hurwitz scheme and, therefore, leads to a computer oriented procedure with superior efficiency over the present techniques for testing positive realness. There is an apparent analogy between the obtained algebraic criteria for positivity (nonnegativity) of a real polynomial and the classical Routh-Hurwitz criteria. Therefore, much remains to be done to exploit this analogy and arrive at new important results.

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SUBOPTIMAL CONTROLS FOR LINEAR REGULATOR SYSTEMS WITH INACCESSIBLE STATES

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Summary

A procedure for designing suboptimal controllers for linear regulator systems is presented. The suboptimal control law is assumed to be a linear function of the system outputs. The form of the time variation for the output-feedback gain matrix is selected arbitrarily, and the free controller parameters are adjusted to minimize the maximum absolute degradation of system performance. An algorithm for computing the controller parameters is discussed, and an extension of the algorithm which corresponds to a modified version of the original problem is proposed.

Introduction

The linear regulator is one of the most extensively studied problems of optimal control theory.[1] The importance of linear regulators stems from the fact that many practical control problems can be formulated in the form of a linear regulator. In addition, the feedback form of the optimal solution to the problem is a very desirable feature. Practical application of the optimal solution to a linear regulator problem suffers from two serious drawbacks: the time-varying nature of the feedback gains, and the need for exact physical measurement of the value of the state vector at every instant of time. In general, not the state but the output vector of the system is available for measurement. Although it is possible to reconstruct the state vector[2], or to obtain an estimate of the state vector that is best in some sense when noise is present , in many cases these schemes are not economically justifiable.

Recently, the design of suboptimal controls which are easy to implement and minimally inferior in some sense to optimal controls has attracted the attention of several investigators. Schoenberger[3] has given solutions to a class of problems in which either the initial state, or a probability distribution of initial conditions is known. Meditch[4] has proposed an approximate method of decoupling complex systems which simplifies the computation of the time-varying feedback gains. Koivuniemi[5] determines the unknown parameters of the feedback control to minimize the maximum degradation of the system performance over the admissible initial states. Rekasius[6] proposes a control law that minimizes the maximum (with respect to all initial states) relative deviation in the value of the performance measure with respect to the optimal performance measure. Another approach is to assume that the admissible initial states are uniformly distributed on the unit hypersphere and determine suboptimal controls that minimize the expected value of the performance measure over the admissible initial states.[7] Salmon[8] suggests that the controller parameters be found by minimaximizing the performance measure or the relative or absolute deviation of the performance measure from its optimal value.

This paper proposes a new technique for determining suboptimal controls for linear regulator systems. The control law is assumed to be linear time-varying feedback of the system <u>outputs</u>. The form of the time variation is selected arbitrarily, and a computational procedure is used to determine the values of the free controller parameters that minimize the maximum absolute degradation of system performance.

The Optimal Linear Regulator Problem

In the optimal linear regulator problem the state and output equations are given by

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t) \tag{1}$$

$$y(t) = C(t)\,x(t) \tag{2}$$

where $x(t)$ is the n-dimensional state vector, $u(t)$ is the m-dimensional unconstrained control vector, $y(t)$ is the r-dimensional output vector, and $A(t)$, $B(t)$, and $C(t)$ are matrices whose elements are assumed to be continuous functions of time. The problem is to determine the control $u^*(\cdot)$ that minimizes the performance measure

$$J(x_o,t_o,u(\cdot)) = \tfrac{1}{2}\,x^T(t_f)\,H\,x(t_f) + \tfrac{1}{2}\int_{t_o}^{t_f}\Big[x^T(t)\,Q(t)\,x(t) + u^T(t)\,R(t)\,u(t)\Big]\,dt \tag{3}$$

where t_o and $t_f > t_o$ are fixed initial and final times, respectively, $x_o = x(t_o)$ is the initial state, H is a real symmetric positive semi-definite matrix, $Q(t)$ is a time-varying real symmetric positive semi-definite matrix, $R(t)$ is a time-varying real symmetric positive definite matrix, and the final state $x(t_f)$ is free. Kalman[1,2] has shown that the optimal control exists and is given by

$$u^*(t) = -R^{-1}(t)\,B^T(t)\,K(t)\,x(t) \triangleq -F^*(t)\,x(t) \tag{4}$$

where $K(t)$ is the unique symmetric positive definite solution of the matrix Riccati equation

$$\dot{K}(t) = -K(t)\,A(t) - A^T(t)\,K(t) - Q(t) + K(t)\,B(t)\,R^{-1}(t)\,B^T(t)\,K(t) \tag{5}$$